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Hereditary Rings Integral over Their Centers

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Bergman [2] has completely characterized the center of a right hereditary ring; the center of a right hereditary ring is a Krull p. p. ring, and any Krull p. p. ring is the center of a right hereditary. While the center of a hereditary ring thus need not be hereditary, there are conditions which imply that the center is hereditary. Robson and Small [13] have shown that the center of a prime *PI* right hereditary ring is a Dedekind domain, and the center of a *PI* hereditary Noetherian ring is a finite direct sum of Dedekind domains. The center of a right hereditary *PI* ring need not be hereditary; Small and Wadsworth [16] have given an example of a *PI* right hereditary, right Noetherian ring whose center is not Noetherian or semihereditary. Jøndrup [11] showed that a right hereditary ring which is module-finite over its center has a hereditary center. Chatters and Jøndrup [5] showed that a PI right hereditary ring which is ring-finite over its center has a hereditary center. We shall prove that any right hereditary ring integral over its center has a hereditary center. Chatters and Jøndrup [5] ask if a right and left hereditary *PI* ring has always has a hereditary center; we will produce an example to show it does not.

We prove several results concerning hereditary rings which are integral over their centers, including a proposition which says that if a right hereditary prime ring Λ is integral over its center R, and if R is not a field, then Λ is a hereditary Noetherian prime ring. This result can be compared to results of Bergman and Cohn [2, Corollary 3.3 and Lemma 1.3] which show that a right hereditary domain whose center is not a field is right Noetherian.

We begin with some examples of hereditary rings Λ which are integral over their centers, but are not included among any of the classes of rings discussed above for which it is known the center is hereditary.

EXAMPLE 1. Let $\Lambda = \bigoplus M_2(Q) + \operatorname{long}\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ be the ring of sequences of 2×2 matrices over the rationals which are eventually some constant lower

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triangular matrix. It is not hard to show that A is a semiprime, hereditary PI ring integral over its center.

The following examples illustrate the extent of the class of prime hereditary rings which are integral over their centers, but not PI. The first of the examples is a hereditary Noetherian prime (HNP) ring which is integral over its center, but is not PI.

EXAMPLE 2. Let D be a division ring with center F which is locally finite, but not PI. Let $A = D[x]$. If $u \in A$, then $u \in E[x]$ where E is a division subring of D which is finite dimensional over F. The ring $E[x]$ is finitely generated, and hence Noetherian as a module, over the Noetherian ring $F[x]$. Since $F[x]$ is the center of A, u is integral over the center of A. The ring A is a hereditary prime Noetherian ring since it is a noncommutative PID. The examples of HNP rings produced by Stafford and Warfield [17, 18], while designed to show that HNP rings can possess certain properties, are integral over their centers and are not PI.

The following example shows that a hereditary ring integral over its center need not have Krull dimension one. It follows from our Theorem 9 and [4, Propositions 1.3 and 1.5] that a hereditary PI ring integral over its center does have Krull dimension one.

EXAMPLE 3. Let A be the infinite alternating group and let $G = A_1 \times A_2 \times \cdots \times A_n$, where A_i is isomorphic to A. Let $A = Q[G]$. Since G is a countable locally finite group, the ring A is a countable regular ring and is hence hereditary. If P_i is the augmentation ideal of the subgroup $A_1 \times \cdots \times A_i$, then $A/P_i \approx Q[A_{i+1} \times \cdots \times A_n]$ is a prime ring. If $P_0 = 0$, then $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n$ is a chain of $n+1$ prime ideals.

The following example is a hereditary prime ring A with Jacobson radical 0 which is algebraic over its center (which is a field), but A is not a regular ring.

EXAMPLES 4. Let A_n be the subring of the ring of linear transformations of a countable dimensional vector space over Q having matrix representation as follows:

$$A_n = \begin{bmatrix} B & & & & \\ & A & & & \\ & & A & & \\ & & & \ddots & \\ & & & & A \end{bmatrix}, \qquad \text{where} \quad B \in M_{2n}(Q)$$

and A is a lower triangular 2×2 matrix. Then $\Lambda = \varinjlim \Lambda_n$ can be shown to have the stated properties.

Our main result follows from a result of Bergman and Cohn [3, p. 89, "some further observations"]; its proof is included for completeness. We thank Professor Bergman for supplying the details of the following two lemmas used to prove the theorem.

LEMMA 5. *Let Λ be a ring containing an invariant non-zerodivisor c, $c\Lambda = \Lambda c$, such that every finitely generated right ideal of Λ containing c is projective. Let I be a projective left ideal of Λ which contains c, and x be any element of Λ. Then the left ideal $I + \Lambda x$ is also projective, and hence any finitely generated left ideal of Λ containing c is projective.*

Proof. Consider the following sequence of left Λ-module homomorphisms:

$$\Lambda \xrightarrow{f} \Lambda \oplus I \xrightarrow{g} \Lambda, \tag{1}$$

where $f(\lambda) = (-\lambda c, \lambda cx)$ and $g(\lambda, i) = \lambda x + i$; note that $cx \in c\Lambda = \Lambda c \subseteq I$, and the composition of the maps is zero. Since I is projective and contains c it can be shown to be finitely generated (as in [9, Lemma 1.2], or see [3, Lemma 1.1]). Applying $* = \mathrm{Hom}(-, \Lambda)$ to (1) produces a sequence of right Λ-module homomorphisms with composition zero:

$$\Lambda \xleftarrow{f^*} \Lambda \oplus I^* \xleftarrow{g^*} \Lambda,$$

where $g^*(\lambda) = (x\lambda, \rho_\lambda)$ for $\rho_\lambda(a) = a\lambda$, and $f^*(\lambda, \phi) = -c\lambda + \phi(cx)$.

The image J of f^* is a finitely generated right ideal of Λ containing c, and hence is projective by the hypothesis. Hence $\Lambda \oplus I^* = J \oplus P$, where P is a right Λ-module complementary summand to the preimage of J in $\Lambda \oplus I^*$ containing $g^*(\Lambda)$. Dualizing again we see that $\Lambda \oplus I = J^* \oplus P^*$, with $f(\Lambda) \subseteq J^*$ and $g(J^*) = 0$. It follows that $I + \Lambda x$, which is $g(\Lambda \oplus I)$, is also $g(P^*)$. Since P and hence P^* are projective, it suffices to show that $g\colon P^* \to \Lambda$ is one-to-one.

Suppose that $(\lambda, i) \in P^* \subseteq \Lambda \oplus I$ is a nonzero element in the kernel of g. Then $(c\lambda, ci) = (\lambda' c, i' c)$ is also a nonzero element in this kernel and in P^*. Now the element $(-c, cx)$ lies in the image of f, hence in J^* which has trivial intersection with P^*, hence $(\lambda' c, i' c) + \lambda'(-c, cx) = (0, i'c + \lambda' cx)$ is another nonzero element of the kernel of g. But $g(0, i'c + \lambda' cx) = 0$ implies that $(0, i'c + \lambda' cx) = (0, 0)$, a contradiction. The final conclusion of the lemma then follows by induction. ∎

LEMMA 6. *Let Λ be a ring satisfying the hypotheses of Lemma 5. Then in* *the ring $\Lambda/c\Lambda$ every finitely generated right ideal is the right annihilator of a finitely generated left ideal (and vice versa).*

Proof. Let I be a finitely generated right ideal of Λ which contains c. Since c is invariant, we can write $cI = Jc$ for some right ideal $J \subseteq R$, which is projective because it is the image of I under an automorphism of Λ, and contains c. As in the proof of the previous lemma, J is finitely generated, say $J = g_1\Lambda + \cdots + g_n\Lambda$; let $f_1, ..., f_n \in \mathrm{Hom}(J, \Lambda)$ be projective coordinates for J. We see that for $y \in J$, we have $y \in Jc$ if and only if $f_i(y) \in \Lambda c$ for all i. Since $c\Lambda \subseteq J$, the maps $y \mapsto f_i(cy)$ $(i = 1, ..., n)$ are defined for all $y \in \Lambda$, and by the above observations, they will simultaneously take values in $c\Lambda$ if and only if $cy \in Jc = cI$; i.e., if and only if $y \in I$. Being right linear maps $\Lambda \to \Lambda$, they can be written $f_i(cy) = a_i y$ for $a_i \in \Lambda$. Then $y \in I$ if and only if $a_i y \in c\Lambda$ for all i. Hence if b_i are the images of a_i in $\Lambda/c\Lambda$, then the image of I in $\Lambda/c\Lambda$ is the right annihilator of the left ideal generated by the b_i. ∎

THEOREM 7 (Bergman and Cohn). *If Λ is right hereditary and c is an invariant non-zerodivisor of Λ then $\Lambda/\Lambda c$ is a right and left Artinian ring.*

Proof. Let $\Lambda_0 = \Lambda/c\Lambda$. Since all right ideals of Λ are projective, as in the proof of Lemma 5 right ideals of Λ containing c are finitely generated, and hence Λ_0 is right Noetherian. By Lemma 6, Λ_0 has DCC on finitely generated left ideals. Hence [1] the Jacobson radical N of Λ_0 is nil and Λ_0/N is right Artinian. Since Λ_0 is right Noetherian N is nilpotent, and hence Λ_0 is right Artinian. Since by Lemma 5 finitely generated left ideals containing c are also projective, Λ_0 is also left Artinian. ∎

PROPOSITION 8. *Let Λ be a right hereditary ring integral over a subring R of the center of Λ, where R is a Krull domain and elements of R are regular in Λ. Then R is a Dedekind domain.*

Proof. By [7, Theorem 43.16] it suffices to show that the Krull dimension of R is 1. Suppose $0 \subsetneq p_1 \subsetneq p_2$ are prime ideals in R. By [4, Propositions 1.2 and 1.3] there exist prime ideals $P_1 \subseteq P_2$ of Λ such that $P_1 \cap R = p_1$ and $P_2 \cap R = p_2$. Take $0 \neq c \in p_1$. Then since $\Lambda/c\Lambda$ is Artinian $P_1 = P_2$, a contradiction. ∎

The proof of the following theorem will make use of the Pierce sheaf representation of a ring Λ. Let $B(\Lambda)$ be the Boolean ring of central idempotents of Λ. The base space of this sheaf representation is $X(\Lambda) = \mathrm{Spec}(B(\Lambda))$, and if $x \in X(\Lambda)$, then $\Lambda_x = \Lambda/x\Lambda$ is the stalk at x. The ring Λ is the ring of global sections of the Pierce sheaf. For further details the reader may consult [12].

THEOREM 9. *Let Λ be a hereditary ring which is integral over its center R. Then R is a hereditary ring.*

Proof. By [3, Corollaries 9.3 and 8.2], R is a Krull p. p. ring with elements of R regular in Λ. Let $x \in X(\Lambda)$; it is enough to show that R_x is a Dedekind domain. Now Λ_x is a hereditary ring which is integral over R_x and R_x is central in Λ_x; moreover nonzero elements of R_x are nonzerodivisors in Λ_x since nonzero elements of R_x can be lifted to regular elements of R (since R is p. p. and hence the support of an element of R is open closed [3, Lemma 3.1]). By Proposition 8, R_x is a Dedekind domain. ▮

When Λ is a right hereditary ring integral over its center, Λ has a total quotient ring Σ with properties we describe in the following result. That Σ need not be a regular ring, even when Λ is prime, can be seen by considering Example 4.

PROPOSITION 10. *Let Λ be integral over its center R and let K be the total quotient ring of R.*

(a) *If Λ is right* p. p. *then Λ is an R-order in $\Sigma = \Lambda \otimes_R K = \Lambda K$ and regular elements of Λ are invertible in Σ.*

(b) *If Λ is right hereditary then*:

(i) *A regular element λ of Λ is contained in only finitely many prime ideals (all of which are maximal ideals).*

(ii) *If I is a right (left) ideal containing a regular element of Λ, then $I \cap R \neq 0$, I is finitely generated, I is an essential right (left) ideal, and I is contained in only finitely many prime ideals (all of which are maximal).*

Proof. (a) When Λ is right p. p., regular elements of R are regular in Λ so that $\Lambda \otimes_R K = \Lambda K$. If λ is regular in Λ, then for any $x \in X(\Lambda)$, λ_x is regular in Λ_x and integral over the field K_x, and hence is invertible in Σ_x. A standard patching argument will produce an inverse for λ in Σ.

(b) (i) The relation $\lambda^{-1} = \gamma r^{-1}$ implies $r = \lambda\gamma$. Hence r is contained in any right ideal containing λ. By Theorem 7 $\Lambda/r\Lambda$ is an Artinian ring, hence it has only finitely many prime ideals, all of which are maximal ideals.

(ii) As in (i) $r = \lambda\gamma$ for r a regular element of R and $\Lambda/r\Lambda$ is an Artinian ring. Hence $0 \neq r \in I \cap R$ and $I/r\Lambda$ is finitely generated so I is finitely generated. If $I \cap A = 0$ for A a right ideal of Λ, then $rA \cap A = 0$ so $rA = 0$ and hence $A = 0$; hence I is an essential right ideal of Λ. As in (i) I can be contained only in the prime ideals of Λ which contain r. ▮

The following proposition shows that much more can be said about the structure of Λ when Λ is integral over a central domain which is not a field. The assumption that the elements of R are regular in the following proposition is necessary. Let $\Gamma = \bigoplus Z_2 \oplus (\text{long } Z_2)$, and let $\Lambda = Z \oplus \Gamma$. If $R = \{(m, \text{long}[m + 2Z])\}$, then $R \approx Z$ and Λ is a hereditary ring which is integral over the central subring R, but Λ is not Noetherian.

PROPOSITION 11. *Let Λ be a right hereditary ring which is integral over a central subring R whose elements are regular in Λ. If R is a Dedekind domain which is not a field, then Λ is a finite direct product of fully bounded (two-sided) hereditary Noetherian prime rings.*

Proof. First, we consider the case where R is a discrete rank one valuation domain with unique maximal ideal M.

By Hoechsmann [10], $M = J(R) = J(\Lambda) \cap R$. Since $J(\Lambda)$ thus contains a regular central element, it follows from Theorem 7 that $\Lambda/J(\Lambda)$ is Artinian. As a result Λ has a complete set of orthogonal idempotents $\{e_1, \ldots, e_n\}$.

Temporarily suppose that Λ is a prime ring. Let $e = e_i$ for some i and consider the ring $e\Lambda e$. Since e is a primitive idempotent, $e\Lambda e$ can contain no nontrivial idempotents. By Sandomierski [14], $e\Lambda e$ is right hereditary; since the annihilator of an element in a hereditary ring is generated by an idempotent, it follows that $e\Lambda e$ is a domain. The ring $e\Lambda e$ is integral over the central subring $eR = eRe$ (which is isomorphic to R) and hence is integral over its center. Since $e\Lambda e$ is a hereditary domain which is integral over its center, Proposition 10 implies that every nonzero right (left) ideal of $e\Lambda e$ is finitely generated and hence $e\Lambda e$ is right and left Noetherian.

Since Λ is a right p. p. ring with no infinite sets of orthogonal idempotents, it is a piecewise domain (PWD) as studied by Gordon and Small [8]. They show that a prime PWD is of the form

$$\Lambda = \begin{bmatrix} D_1 & \cdots & D_{1r} \\ \vdots & D_2 \quad \ddots & \vdots \\ D_{r1} & \cdots & D_r \end{bmatrix},$$

where each $D_i = e_i \Lambda e_i$ is a domain and each D_{jk} is isomorphic as a right D_k-module to a nonzero right ideal in D_k and as a left D_j-module to a nonzero left ideal in D_j. Since the $D_i = e_i \Lambda e_i$ are right hereditary domains [14] integral over $e_i R$ and hence right and left Noetherian by the above argument, it then follows that Λ is right and left Noetherian.

Now consider the case in which Λ is not necessarily prime. Since Λ is a

PWD it again follows from Gordon and Small [8] that Λ has a triangular structure as in the figure below.

$$\Lambda \approx \begin{bmatrix} P_1 & & & \\ P_{21} & P_2 & & \\ \vdots & \vdots & \ddots & \\ P_{n1} & P_{n2} & \cdots & P_n \end{bmatrix}.$$

Each diagonal ring P_i is prime and each P_{ij} is a $P_i - P_j$ bimodule. If f_i is an idempotent of Λ such that $P_i = f_i \Lambda f_i$, then P_i will be a right hereditary prime ring which is integral over the discrete valuation domain $f_i R f_i$. By the above P_i is Noetherian. Let Γ be the localization of Λ at the nonzero elements of R, and let F be the quotient field of R. The nonzero elements of R are central regular elements of Λ so that we can consider Λ as a subring of Γ; this also implies that if $f_i \Gamma f_j$ is nonzero then so is $f_i \Lambda f_j$. Hence Γ has a triangular structure

$$\Gamma \approx \begin{bmatrix} K_1 & & & \\ K_{21} & K_2 & & \\ \vdots & \vdots & \ddots & \\ K_{n1} & K_{n2} & \cdots & K_n \end{bmatrix}.$$

Each K_i is the localization of the Noetherian prime ring P_i at the set of nonzero elements of $f_i R$ and hence is a prime Goldie ring which is integral over the field $f_i F$; it is easy to see that K_i is thus equal to its classical quotient ring and is hence simple Artinian. Consequently Γ is a semiprimary two-sided classical quotient ring of Λ. Suppose that P_{ij} is nonzero. It is not difficult to see that P_{ij} must be a projective right P_j-module. It follows from Small [15] that Λ is left semihereditary since Λ does not contain any infinite sets of orthogonal idempotents. By Theorem 3.11 of [6], $P_{ij} = K_{ij}$. Since K_j is simple Artinian, this means that K_j is projective as a right P_j-module, which by Lemma 3.6 of [6] cannot be the case unless $P_j = K_j$. Hence the simple Artinian ring K_j is integral over the central domain $f_j R$; the standard commutative proof then shows that $f_j R$ is a field. This is a contradiction, for $f_j R$ is isomorphic to R which is not a field. Hence it must be the case that each $P_{ij} = 0$, and Λ is a semiprime Noetherian hereditary ring.

Now consider the case where R is a general Dedekind domain which is not a field. If P is a nonzero prime ideal of R, then Λ_P will be a hereditary ring which is integral over the discrete valuation domain R_p. Since the nonzero elements of R are regular in Λ, the nonzero elements of R_p are regular in Λ_p and we can identify Λ with a subring of Λ_p. The above shows that Λ_p

is semiprime Goldie, and hence Λ has a semisimple Artinian quotient ring which is obtained by inverting elements of R by Proposition 10. By Goldie's Theorem, Λ is semiprime. If I is an essential right (left) ideal of Λ, then I must contain a regular element and hence by Proposition 10 I is finitely generated and contains a regular element of R. Hence Λ is a bounded two-sided Noetherian hereditary semiprime ring. If P is a prime ideal in an HNP summand Λ_i, then Λ_i/P is Artinian and hence Λ_i is fully bounded. ▮

COROLLARY 12. *Let Λ be a hereditary prime ring which is integral over its center R. If R is not a field, then Λ is a fully bounded hereditary Noetherian prime ring.*

COROLLARY 13. *Let Λ be a hereditary ring which is integral over its center R. If $x \in X(\Lambda)$ is such that R_x is not a field, then Λ_x is a Noetherian semiprime ring.*

Both Examples 3 and 4 show that the fact that R is a Dedekind domain which is not a field is necessary in Proposition 11. Example 2 shows that under the hypothesis of Corollary 12 the ring Λ, while Noetherian, need not be ring-finite over its center. The following example shows that the assumption that Λ is integral over its center is necessary in Corollary 12.

EXAMPLE 14. Let R be a *PID* (which is not a field) and K be its quotient field. Consider the subring Λ_n of the ring of linear transformations of a countable vector space over K having the following matrix representation:

$$\Lambda_n = \begin{bmatrix} B & & & \\ & a & & \\ & & \ddots & \\ & & & a \end{bmatrix}, \qquad \text{where} \quad B \in M_n(K)$$

and $a \in R$. Then $\Lambda = \varinjlim \Lambda_n$ is a hereditary prime non-Noetherian ring with center R, a Dedekind domain which is not a field.

We conclude with an example of a *PI* ring Λ which is right and left hereditary, semiprime, possesses a regular quotient ring obtained by inverting central regular elements, yet $Z(\Lambda)$ is not hereditary. This answers a question of Chatters and Jøndrup [11].

EXAMPLE 15. Let F be a field and let $E = F(x, y)$ be the rational functions over F in two indeterminates. Let $\Lambda = (\bigoplus M_2(E)) \oplus \operatorname{long}\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ where $a \in F(x)[y]$ and $b \in F(y)[x]$; i.e., Λ is sequences of 2×2 matrices over E which eventually are a diagonal matrix of the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Clearly Λ is a semiprime PI ring with regular total quotient ring $\Sigma = \bigoplus M_2(E) \oplus \text{long}\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$ with $c, d \in F(x, y)$. One can show that $Z(\Lambda) \cong \bigoplus E + \text{long } F[x, y]$ (i.e., $Z(\Lambda)$ is all sequences of scalar matrices in E which eventually are in $F[x, y]$) because $F(x)[y] \cap F(y)[x] = F[x, y]$. Clearly $\bigoplus E + \text{long } F[x, y]$ has global dimension at least 2.

To show that Λ is hereditary, it suffices by symmetry to show that it is right hereditary. It is easy to see that $X(\Lambda) = \{1, 2, ...\} \cup \{\infty\}$, the one point compactification of the positive integers. Let $e(n)$ be the central idempotent of Λ such that $e(n)_n = 1_n$ and $e(n)_x = 0_x$ for all x not equal to n. Let I be a right ideal of Λ. Since Λ_∞ is a PIR, $I_\infty = (a\Lambda)_\infty$ for some $a \in I$. Since $e(n)\Lambda \approx M_2(E)$, there exists a semisimple right submodule $J(n)$ of $e(n)I$ such that $e(n)(a\Lambda) \oplus J(n) = e(n)I$. Let $J = \bigoplus_n J(n)$; J is projective since it is a direct sum of direct summands of Λ. It is easily verified that $I = J \oplus a\Lambda$. To show that I is projective, we can assume without loss of generality that $I = a\Lambda$ where a is a constant sequence of a fixed diagonal matrix. By considering the four possible arrangements of nonzero entries in a, one sees that in each case the idempotent generating the right annihilator of a in $\Lambda_x = M_2(E)$ is the same as the idempotent generating the right annihilator of a in $\Lambda_\infty = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$ (e.g., rt $\text{annih}_{\Lambda_x}\begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}$ and rt $\text{annih}_{\Lambda_\infty}\begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}$ are both generated by $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$). Hence rt $\text{annih}_\Lambda(a\Lambda) = e\Lambda$ for an idempotent e of Λ, and hence $a\Lambda$ is Λ-projective.

We note that the Krull dimension of Λ is 1 and the Krull dimension of $Z(\Lambda)$ is 2; by using $n \times n$ matrices and n indeterminates, similar examples with centers of Krull dimension n can be produced. A more complicated construction using double sequences yields an example of a semiprime hereditary ring of PI degree 2 with a center of infinite Krull dimension. We know of no example of a semiprime hereditary PI ring which does not have Krull dimension one.

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